Continuous logic and its fragments

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First order logic:
Reasoning on discrete objects $x, y, ...$ and relations between them.

Ex. ‘socrates is a Human’ : H(s)

‘plato is Student of socrates’ p S s

H: mans $\to \{0, 1\}$,
S: mans $\times$ mans $\to \{0, 1\}$

quantifiers may be used: $\forall x \exists y \ \ x\mathsf{S}y$

A more abstract universe is $(\mathbb{N}, <)$
e.g. $1 < 2$

More generally: orders, graphs, groups, rings, ...

Ex.

$$\mathbb{N} \models [\phi(0) \wedge \forall x(\phi(x) \to \phi(x+1))] \to \forall x \phi(x)$$

$$G \models \forall x \exists y_1 ... y_n \bigwedge_i xRy_i$$

FOL $\approx$ Logical study of first order structures (no continuity on structures)

Continuous logic:
Reasoning on continuous objects:
first order structures equipped with a notion of distance (topology)

Ex. Let the universe of discourse be the earth $= \mathbb{S}^2$

$$d(x, y) \leqslant 4000km$$

$$h(e) \geqslant 8000m$$

$$B(x) = 100mm$$

$$\int_{Iran} B(x) = 50$$

More generally: Metric structures: Riemannian manifolds (say Minkowski space), $\mathbb{R}^n$, $\mathbb{H}$, $\mathbf{C}(X)$, ...

Interested statements:
$d(x,y) \leqslant r$, $\sup_x h(x) = 8882$,
$f(x) = g(x)$

instances of proofs:

$$\frac{f \leqslant g, \quad g \leqslant h}{f \leqslant h}, \qquad \frac{f = g}{r \cdot f = r \cdot g}$$

$$\frac{f(0) = 0, \quad f' \geqslant 0}{f \geqslant 0}, \qquad \frac{f \geqslant 0}{\int f \geqslant 0}$$

Question

How to construct a logic for reasoning on these statements?

s.t. proofs be continuous:
if $\frac{\phi \leqslant 0}{\psi \leqslant 0}$ then $\forall \epsilon \; \exists \delta \quad \frac{\phi \leqslant \delta}{\psi \leqslant \epsilon}$

The algebra of truth values in FOL:

$\Big(\{0,1\}, \wedge, \vee, \neg, \ \exists, \forall\Big)$

(Note: $\exists = \sup$, $\forall = \inf$)

Equivalently: $(\mathbb{Z}_2, \oplus, \otimes, \quad \sup, \inf)$

More generally: A value structure is like

$(A,$ connectives, quantifiers$)$

Ex. $(\mathbb{R}, +, -, \quad \sup, \inf)$

Definition:

Let $\mathbb{V}$ be a value space
(Chang-Keisler: a Compact Hausdorff space)

connective: $\alpha : \mathbb{V}^n \to \mathbb{V}$
(Chang-Keisler: Continuous)
Ex. $\alpha(\phi_1, \phi_2) = \text{min}\{v(\phi_1), v(\phi_2)\}$
$(= \phi_1 \wedge \phi_2)$

quantifier $Q : \mathscr{P}(\mathbb{V}) \to \mathbb{V}$
(Chang-Keisler: continuous)
Ex. $A \subseteq \mathbb{V}$, $\qquad A \mapsto \text{sup}(A)$
$\text{sup}_x \phi(x) = \text{sup}\{v(\phi(a))|\ a \in M\}$

FOL is the logic based on the value structure
$\Big(\{0,1\},\ \wedge,\neg,\quad \exists\Big)$

Question: What is the logic based on the value structure

$$\Big(\mathbb{A},+,\times,\wedge,\vee,\ldots;\ \mathsf{sup},\mathsf{inf},\quad =,\leqslant\Big)$$

or its reductions

$\mathbb{A}=\mathbb{Z},\ \mathbb{R},\ \mathbb{R}^n,...$

Ex. imagine a vector $v(\phi)=\Big(\phi^m,\phi^t,\phi^h\Big)$

some references:

- C.C. Chang, J.H. Keisler, *Continuous model theory*, Princeton University Press (1966)

- J. Flum, M. Ziegler, *Topological model theory*, LNM, Springer-Verlag (1980)

- W. Henson, J. Iovino, *Ultraproducts in analysis*, in Analysis and Logic, v. 262, London Mathematical Society lecture notes (2002)

Modern continuous logic is based on

$$\Big([a,b],\ \wedge,\ \vee,\ b-x,\ \ \mathsf{sup}, \mathsf{inf},\ \ \leq\Big)$$

or equivalently $(\mathbb{R}, +,\ \ \vee, \wedge,\ \ \mathsf{sup})$
or even $(\mathbb{R}, +, \times,\ \ \mathsf{sup})$

Language: $\mathscr{L} = \{F, ...;\ P, ...; , c, ..\} \cup \{d\}$
Formulas:
$P(\bar{x})$, $\phi + \psi$, $\phi \cdot \psi$, $\phi \wedge \psi$, $\mathsf{sup}_x\, \phi$, ...

Statements $\phi = r$, $\phi \geqslant 0$, ...

Ex. $d(x,z) \leqslant d(x,y) + d(y,z)$
$d(x,y) \leqslant 4000$, $\mathsf{sup}_{xy}\, d(x,y) = 2$

What holds in continuous logic:

compactness, downward, upward, e.chain, e.j.e., e.a.p., all definability theoerms (Beth, Svenonus, Herbrand, Robinson, ...), types, saturation, categoricity, stability, Fraisse construction, ...

Th. The theory of Hilbert spaces is totally categorical and ω-stable

non-forking = orthogonality

Theorem Continuous logic is a conservative extension of first order logic:

$$\text{FOL} \subsetneqq \text{CL}$$

CL is very strong on compact models:

Theorem If M is compact, all its closed subsets are definable

Reference:
Ben-Yaacov, Berenstein, Henson, Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis, volume 2 (Zoe Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.),

Linear continuous logic:
$(\mathbb{R}, +, -, \quad \text{sup}, \text{inf}, \quad =, \leqslant)$

Formulas: atomics, $\phi + \psi$, $\phi - \psi$, $\text{sup}_x \phi$
Statements: $\phi = r$, $\phi = \psi$, $\phi \leqslant \psi$

Theorem A set T of conditions $\phi \geqslant$ is satisfiable iff every linear combination $\sum_i \alpha_i \phi_i \geqslant 0$ is satisfiable.
Corollary: Robinson, Herbrand, ...
Upward: Any M with $|M| \geqslant 2$ has arbitrarily large elementary extensions
Corollary Linear CL $<$ CL

However:
if $m \neq n$ then $\mathbb{S}^m \not\equiv \mathbb{S}^n$
M, N finite f.o., $M \equiv N$ implies $M \simeq N$

Ex: $(\mathbb{Z}_2, d) \equiv (2^\omega, d) \equiv 2^\lambda$

$d(\bar{a}, \bar{b}) = \sum_i \frac{|x-y|}{2^{i+1}}$
λ Lebesgue measure

Th. $Th(\mathbb{Z}_2, +)$ has QE, is ω-stable.

Is it categorical?

Question: Axiomatize the theory of $(\mathbb{S}^2)$.

Th. If M is compact then $X \subseteq M$ is definable iff there exists $\phi(x)$ and r such that

$$\sup_x \phi^M(x) = r,$$

$$X = \{a \in M \mid \phi^M(a) = r\}$$

Examples of definable sets:

- In $\mathbb{S}^2$ every point is defined by means of its antipode: take $d(a, x)$
- The arc between x, y in $\mathbb{S}^2$
- (M, f) a dynamical system: per(f), closure of the set of periodic points
- if G is a compact group, Tor(G)
- closed unit disc, the boundary as well as the center are definable

CL versus linear CL:

CL	linear CL
compactness	linear compactness
acl - dcl ✓	dcl ✓
compact $\approx$ finite	-
type = character	type=functional=measure
$S_n(T)$ compact	$S_n(T)$ compact convex
-	extreme types are principal

Questions:

- What is the possible linearization of FOL?

- Expand CL with other operators integration, derivation,...