Friedman's Theorem: from standard systems to fixed points

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Peano Arithmetic (1889)

$\mathcal{L}_A := \{0, 1, +, . , <\}$
PA^- := the positive segment of discretely ordered rings; i.e.

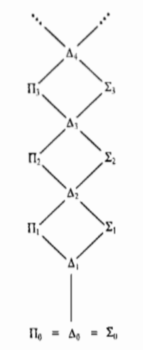
- associativity of . and +
- distributivity of . on +
- commutativity of . and +
- $\forall x\ (x + 0 = x)$, $\forall x\ (x.0 = 0)$ and $\forall x\ (x.1 = x)$
- $<$ is linearly ordered
- $\forall x, y, z\ (x < y \rightarrow x + z < y + z)$ and
  $\forall x, y, z\ ((z > 0\ \wedge\ x < y) \rightarrow x.z < y.z)$

$\mathrm{PA} := \mathrm{PA}^- +\ \mathrm{I}\varphi$; for every $\varphi \in \mathrm{Form}(\mathcal{L}_A)$.
$\mathrm{I}\varphi$:

$$\forall z\ [(\varphi(0, z) \wedge (\forall x\ (\varphi(x, z) \rightarrow \varphi(x + 1, z)))) \rightarrow \forall x\ (\varphi(x, z))].$$

Fragments of Arithmetic

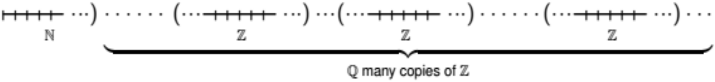
- $\Sigma_0 = \Pi_0 = \Delta_0 :=$ the class of all bounded formulas; i.e those formulas whose quantifiers all occur as $\forall x < t$ or $\exists x < t$.
- $\Sigma_{n+1} = \{\exists x \varphi(x) : \varphi \in \Pi_n\}$;
  $\Pi_{n+1} = \{\forall x \varphi(x) : \varphi \in \Sigma_n\}$.



- $\mathrm{I}\Sigma_n = \mathrm{PA}^- + \mathrm{I}\varphi$, for every $\varphi \in \Sigma_n$;
  $\mathrm{I}\Pi_n = \mathrm{PA}^- + \mathrm{I}\varphi$, for every $\varphi \in \Pi_n$.
- $\mathrm{Exp} := \ \forall x \ \exists y \ (y = 2^x)$.

Models of Arithmetic

- **Standrad model**: $\mathbb{N} := (\omega, 0, 1, +, . , <)$.
- **Nonstandard models**: Tarski (1934)



- $\mathrm{I} \subseteq \mathrm{M}$ is an **initail segment** if for all $a, b \in \mathrm{M}$ if $a < b$ and $b \in \mathrm{I}$, then $a \in \mathrm{I}$.
- An initial segment with no maximum element in $\mathcal{M}$ is called a **cut**.
- **Σ_n-Overspill.** For every Σ_n-formula $\varphi(x)$ and every proper cut I of model $\mathcal{M}$ of $\mathrm{I}\Sigma_n$, if φ holds for every element of I, then there is some $a > \mathrm{I}$ s.t. φ holds for every element of M which is less than a.

Coding in $\mathcal{M} \models \mathrm{I}\Delta_0 + \mathrm{Exp}$:

For all $a \in \mathrm{M}$ there exists unique $c < a$ and $a_0 < ... < a_{c-1} < a$ such that the sequence $(a_0, ..., a_{c-1})$ is the binary expansion of a in $\mathcal{M}$.
We say $x \mathrm{E} a$ iff there is some $y < c$ s.t. $x = a_y$.
$a_{\mathrm{E}} := \{x \in \mathrm{M} : x\mathrm{E}a\}$.
$\mathrm{X} \subseteq \mathrm{M}$ is **coded** if there is some $a \in \mathrm{M}$ s.t. $\mathrm{X} = a_{\mathrm{E}}$.

Satisfaction Predicate

$(n \in \omega)$ Sat_{Σ_n} is the $\mathcal{L}_{\mathrm{A}}$-formula defining the satisfaction predicate for Σ_n-formulas for an ambient model satisfying $\mathrm{I}\Delta_0 + \mathrm{Exp}$.

Types

- A **type** over $\mathcal{M}$ is a family $p(x)$ of formulas with free variable x and finitely many parameters in $\mathcal{M}$ which is finitely satisfiable in $\mathcal{M}$.
- A type is Σ_n, if all of its elements are Σ_n-formulas.
- $p(x)$ is a recursive type, if $\{\ulcorner \varphi(x,y) \urcorner : \varphi(x,a) \in p(x)\}$ is a recursive subset of ω.
- Model $\mathcal{M}$ is **recursively saturated** if all recursive types are realized in $\mathcal{M}$.
- Every nonstandard model of PA is Σ_n-recursively saturated, for all $n \in \omega$.
  More generally, every coded Σ_n-type is realized in a nonstandard model of $I\Sigma_n$.

Scott's sets

Let T be a completion of PA. We say $X \subseteq \omega$ is **representable** in T if there exists some formula φ such that:

$$n \in X \text{ iff } T \vdash \varphi(n).$$

$$\mathrm{Rep}(T) := \text{the family of all representable subsets of } \omega \text{ in } T.$$

Definition

Scott set $\mathcal{X} \subseteq \mathrm{P}(\omega)$ *is called a Scott set iff:*

1) $\mathcal{X}$ *is a Boolean algebra of sets.*
2) $\mathcal{X}$ *is closed under recursion; i.e. if* $A \in \mathcal{X}$ *and* B *is recursive over* A*, then* $B \in \mathcal{X}$.
3) *Weak König's Lemma holds for* $\mathcal{X}$*; i.e. if* $\mathrm{Tr} \in \mathcal{X}$ *is an infinite binary tree, then there is some* $P \in \mathcal{X}$ *s.t.* P *is an infinite branch of* Tr.

Theorem (Scott 1962)

Let T *be an axiomatizable theory containing* PA *and* $\mathcal{X}$ *be a countable Scott set s.t.* $\mathrm{T} \in \mathcal{X}$*. Then there are continuum many completions* $\overline{\mathrm{T}}$ *of* T *s.t.* $\mathcal{X} = \mathrm{Rep}(\overline{\mathrm{T}})$*.*

Standard Systems (Friedman 1973)

Definition

$\mathrm{SSy}(\mathcal{M}) := \{\omega \cap a_{\mathrm{E}} : a \in \mathrm{M}\}$.

Theorem (Corollary of Scott's Theorem)

1) *For each model* $\mathcal{M}$ *of* PA, $\mathrm{SSy}(\mathcal{M})$ *is a Scott set.*
2) *If* $\mathcal{X}$ *is a countable Scott set, then there is a model* $\mathcal{M}$ *of* PA *such that* $\mathrm{SSy}(\mathcal{M}) = \mathcal{X}$*.*

Lemma

For every nonstandard model $\mathcal{M}$ of PA it holds that:

1) $\mathrm{SSy}(\mathcal{M}) = \{\omega \cap \mathrm{X} : \ \mathrm{X} \text{ is definable in } \mathcal{M}\}$.
2) *If $a \in \mathrm{M}$, then $tp_{\Sigma_n}(a) \in \mathrm{SSy}(\mathcal{M})$.*
   In particular, if $\mathcal{M}$ is recursively saturated $tp(a) \in \mathrm{SSy}(\mathcal{M})$.

A very unstable theory

- $\mathrm{Th}(\mathcal{M})$
- $S(\mathcal{M}) \qquad \rightsquigarrow \qquad \mathrm{SSy}(\mathcal{M})$
- $\mathrm{Lt}(\mathcal{M})$

Recursively Saturated Models (1970s)

Theorem

Let $\mathcal{M} \models \mathrm{PA}$*. Then the following are equivalent:*

1) $\mathcal{M}$ *is recursively saturated.*
2) $\mathcal{M}$ *is* $\mathrm{SSy}(\mathcal{M})$*-saturated.*
3) *For all* $a \in \mathrm{M}$, $tp(a) \in \mathrm{SSy}(\mathcal{M})$.

Theorem

Let $\mathcal{M}$ *and* N *be countable recursively saturated models of* PA*. Then* $\mathcal{M} \cong \mathcal{N}$ *iff* $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$ *and* $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$.

Scott (1950s)

Is there a model of PA that is isomorphic to a proper initial segment of itself?

Vaught (1962)

There is a model of true arithmetic that is isomorphic to a proper initial segment of itself.

Friedman (1973)

Let $\mathcal{M}$, $\mathcal{N}$ be countable nonstandrd models of PA. The following statements are equivalent:

1. $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$, and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}) \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{N})$.
2. There is an embedding $j : \mathcal{M} \to \mathcal{N}$ such that $j(\mathrm{M}) \subsetneq_e \mathrm{N}$.

Wilkie (1977)

There are continuum-many initial segments of every countable nonstandard model of $\mathcal{M}$ of PA that are isomorphic to $\mathcal{M}$.

Wilkie (1977)

If $\mathcal{M}$ and $\mathcal{N}$ are countable nonstandard models of PA, then there are arbitrarily high initial segment of $\mathcal{N}$ that are isomorphic to $\mathcal{M}$ iff $\mathrm{SSy}(\mathcal{M}) = \mathrm{SSy}(\mathcal{N})$ and $\mathrm{Th}_{\Pi_2}(\mathcal{M}) \subseteq \mathrm{Th}_{\Pi_2}(\mathcal{N})$.

Lipshitz (1979)

A countable nonstandard model $\mathcal{M}$ of PA can be embedded into arbitrarily low nonstandard initial segments of itself iff $\mathcal{M} \models \mathrm{Th}_{\Pi_1}(\mathbb{N})$.

Solovay (1981)

Every countable **recursively saturated** model of $I\Delta_0 + B\Sigma_1$ is isomorphic to a proper initial segment of itself.

Ressayre (1987)

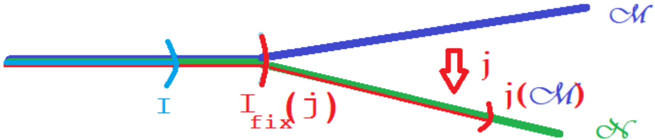
For every countable nonstandard model $\mathcal{M}$ of $I\Delta_0$, $\mathcal{M} \models I\Sigma_1$ iff for every $a \in \mathrm{M}$ there is a proper initial self-embedding j of $\mathcal{M}$ such that $j(x) = x$ for all $x \leq a$.

Generalization of SSy($\mathcal{M}$)

Let I be a cut in $\mathcal{M}$.
$\mathrm{SSy}_{\mathrm{I}}(\mathcal{M}) := \{\mathrm{I} \cap a_{\mathrm{E}} :\ a \in \mathrm{M} \setminus \mathrm{I}\}$.

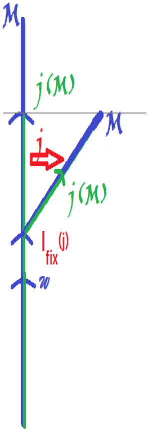
Hájek & Pudlák (1980)

If I is a cut closed under exponentiation that is shared by two nonstandard models $\mathcal{M}$ and $\mathcal{N}$ of PA such that $\mathcal{M}$ and $\mathcal{N}$ have the same I-standard system, and $\mathrm{Th}_{\Sigma_1}(\mathcal{M}, i)_{i \in I} \subseteq \mathrm{Th}_{\Sigma_1}(\mathcal{N}, i)_{i \in I}$, then there is an embedding j of $\mathcal{M}$ onto a proper initial segment of $\mathcal{N}$ such that $j(i) = i$ for all $i \in I$.



Fixed points

Let $j : \mathcal{M} \to \mathcal{M}$ is a self-embedding:



- $\mathrm{I}_{\mathrm{fix}}(j) := \{m \in M : \forall x \le m \; j(x) = x\}$.
- $\mathrm{Fix}(j) := \{m \in M : j(m) = m\}$.

Automorphisms (1990s)

Suppose $\mathcal{M}$ is a countable **recursively saturated** model of PA, and I is a proper initial segment of $\mathcal{M}$.

(a) (**Smoryński**) $\mathrm{I} = \mathrm{I}_{\text{fix}}(j)$ for some automorphism j of $\mathcal{M}$ iff I is closed under exponentiation.

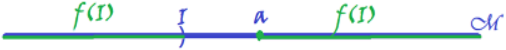
(b) (**Kaye-Kossak-Kotlarski**) $\mathrm{I} = \mathrm{Fix}(j)$ for some automorphism j of $\mathcal{M}$ iff I is a strong cut of $\mathcal{M}$ and $\mathrm{I} \prec \mathcal{M}$.

(c) (**Kaye-Kossak-Kotlarski**) $\mathrm{Fix}(j) = K(\mathcal{M})$ for some automorphism j of $\mathcal{M}$ (i.e. j moves every undefinable element of $\mathcal{M}$) iff $\mathbb{N}$ is a strong cut of $\mathcal{M}$.

Strong Cuts (Kirby-Paris 1977)

Definition

Given a cut I of $\mathcal{M}$, I is said to be a strong cut of $\mathcal{M}$ if, for each function f whose graph is coded in $\mathcal{M}$ and whose domain includes I, there is some a in M such that for all $m \in I$, $f(m) \notin I$ iff $a < f(m)$.



Theorem (Kirby-Paris (1977))

Let $\mathcal{M} \models \mathrm{PA}$ *and* I *be a proper cut of* $\mathcal{M}$. *Then the following are equivalent:*

1) I *is strong in* $\mathcal{M}$.
2) $(\mathrm{I}, \mathrm{SSy}_{\mathrm{I}}(\mathcal{M})) \models \mathrm{ACA}_0$; *i.e. i) for all* $\mathrm{X} \in \mathrm{SSy}_{\mathrm{I}}(\mathcal{M})$, $(\mathcal{M}, \mathrm{X}) \models \mathrm{PA}^*$ *and, ii)* $\mathrm{SSy}_{\mathrm{I}}(\mathcal{M})$ *is closed under arithmetical comprehension.*
3) *For all* $a \in \mathrm{I}$ *and* $n \in \omega$, $\mathrm{I} \longrightarrow (\mathrm{I})^n_a$; *i.e. for every coded function* $f : [\mathrm{I}]^n \longrightarrow a$, *there is some* $\mathrm{A} \in \mathrm{SSy}_{\mathrm{I}}(\mathcal{M})$ *s.t.* f *is constant on* $[\mathrm{A}]^n$.

Theorem

1) *There exists model* $\mathcal{M}$ *of* PA *in which* $\mathbb{N}$ *is a strong cut.*
2) *There exists model* $\mathcal{N}$ *of* PA *in which* $\mathbb{N}$ *is not a strong cut.*

Bahrami-Enayat (2018)

Suppose $\mathcal{M}$ is a countable model of $\mathrm{I}\Sigma_1$, and I is a proper cut of $\mathcal{M}$. Then:

1) $\mathrm{I} = \mathrm{I}_{\mathrm{fix}}(j)$ for some proper initial self-embedding j of $\mathcal{M}$ iff I is closed under exponentiation.
2) $\mathrm{I} = \mathrm{Fix}(j)$ for some proper initial self-embedding j of $\mathcal{M}$ iff I is a strong cut of $\mathcal{M}$ and $\mathrm{I} \prec_{\Sigma_1} \mathcal{M}$.
3) $\mathrm{Fix}(j) = K^1(\mathcal{M})$ for some proper initial self-embedding j of $\mathcal{M}$ (i.e. j moves every Σ_1-undefinable element of $\mathcal{M}$) iff $\mathbb{N}$ is a strong cut of $\mathcal{M}$.

WKL_0

$(\mathcal{M}, \mathcal{A}) \models \mathrm{WKL}_0$ iff

1) $(\mathcal{M}, S)_{S \in \mathcal{A}} \models \mathrm{I}\Sigma_1$
2) Comprehension for Δ^0_1-formulas
3) Weak König's Lemma (which asserts that every infinite subtree of the full binary tree has an infinite branch)

Bahrami (2019)

Suppose $(\mathcal{M}, \mathcal{A})$ is a countable model of WKL_0, and I is a proper cut of $\mathcal{M}$. Then:

1) $\mathrm{I} = \mathrm{I}_{\mathrm{fix}}(j)$ for some proper initial self-embedding j of $(\mathcal{M}, \mathcal{A})$ iff I is closed under exponentiation.
2) $\mathrm{I} = \mathrm{Fix}(j)$ for some proper initial self-embedding j of $(\mathcal{M}, \mathcal{A})$ iff I is a strong cut of $\mathcal{M}$ and $\mathrm{I} \prec_{\Sigma_1} \mathcal{M}$.

Thank you!