Intuitionistic analogues of the Łos-Tarski Theorem

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- Intuitionistic Universal and Existential Closures
- Generalized Preservation Theorems
- Chains of Kripke Models

In classical model theory, much attention has been devoted to characterizing the connection between classes of models and their first order syntactic descriptions. The most well-known characterization of this sort is Godel's completeness theorem. Other wellknown characterizations are the syntactic preservation theorems of classical model theory. The Łos-Tarski Theorem states that a classical theory is axiomatizable by universal sentences if and only if it is preserved under submodels.

The Lyndon- Łos-Tarski Theorem (sometimes called the homomorphism preservation theorem) states that a classical theory is axiomatizable by existential-positive sentences if and only if it is preserved under homomorphisms of models. The Chang- Łos-Suszko Theorem and Keisler Sandwich Theorem state that a classical theory is axiomatizable by universal-existential sentences if and only if it is preserved under unions of chains of models if and only if it is preserved under sandwiches of models.

The Lyndon- Łos-Tarski Theorem (sometimes called the homomorphism preservation theorem) states that a classical theory is axiomatizable by existential-positive sentences if and only if it is preserved under homomorphisms of models. The Chang- Łos-Suszko Theorem and Keisler Sandwich Theorem state that a classical theory is axiomatizable by universal-existential sentences if and only if it is preserved under unions of chains of models if and only if it is preserved under sandwiches of models.

In this talk, we investigate intuitionistic analogues of Łos-Tarski Theorem in the context of Kripke models.

Basic Definitions

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Fact (Łos-Tarski Theorem)

Let T be a classical theory in $\mathcal{L}$. Then:

- *T is preserved under taking submodels if and only if T is axiomatizable by $\forall_1$-sentences.*
- *T is preserved under taking extensions if and only if T is axiomatizable by $\exists_1$-sentences.*

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A natural question is what the analogue of this theorem in the context of Kripke models is?

Definition

A Kripke model $\mathfrak{A}$ for the language $\mathcal{L}$, is an ordered pair $\mathfrak{A} = ((A_\alpha)_{\alpha \in F}, \leq)$ such that:

- $(F, \leq)$ is a partially ordered set (called the frame of $\mathfrak{A}$),
- to each element (called a node) α of F is attached a classical structure A_α such that: $\alpha \leq \beta \Rightarrow A_\alpha \subseteq_w A_\beta$ (weak substructure).

Basic Definitions

Definition

The Forcing relation $\Vdash$ is defined inductively as follows (where φ, ψ are $\mathcal{L}_\alpha$-sentences):

- For atomic φ, $\alpha \Vdash \varphi$ if and only if $A_\alpha \vDash \varphi$, also, $\alpha \nVdash \bot$;
- $\alpha \Vdash \varphi \vee \psi$ if and only if $\alpha \Vdash \varphi$ or $\alpha \Vdash \psi$;
- $\alpha \Vdash \varphi \wedge \psi$ if and only if $\alpha \Vdash \varphi$ and $\alpha \Vdash \psi$;
- $\alpha \Vdash \varphi \to \psi$ if and only if for all $\beta \geq \alpha$, $\beta \Vdash \varphi$ implies $\beta \Vdash \psi$;
- $\alpha \Vdash \forall x \varphi(x)$ if and only if for all $\beta \geq \alpha$ and all $a \in A_\beta$, $\beta \Vdash \varphi(a)$;
- $\alpha \Vdash \exists x \varphi(x)$ if and only if there exists $a \in A_\alpha$ such that $\alpha \Vdash \varphi(a)$.

There are several ways to define the notion of submodel for Kripke models of intuitionistic first-order logic: one might restricts the frame, or the classical structures attached to the nodes, or both.

In [V], [MZ1] and [EFMR], the authors choose the first, second and third definition of submodel, respectively, and characterize theories that are preserved under taking submodels.

In [MZ1], theories that are preserved under taking extensions for the second definition of submodel are also characterized.

In [Z], theories that are preserved under taking extensions for the first and third definition of submodel are characterized.

Definition

Let $\mathfrak{A} = ((A_\alpha)_{\alpha \in \boldsymbol{A}}, \leq_{\boldsymbol{A}})$ and $\mathfrak{B} = ((B_\alpha)_{\alpha \in \boldsymbol{B}}, \leq_{\boldsymbol{B}})$ be Kripke models. Then $\mathfrak{A}$ is a **submodel** of $\mathfrak{B}$, written $\mathfrak{A} \subseteq^1 \mathfrak{B}$, if and only if

1. $\boldsymbol{A}$ is a subset of $\boldsymbol{B}$ and $\leq_{\boldsymbol{A}} = \leq_{\boldsymbol{B}}\restriction \boldsymbol{A}$,
2. For all $\alpha \in \boldsymbol{A}$, $A_\alpha = B_\alpha$.

We also say that $\mathfrak{B}$ is an **extension** of $\mathfrak{A}$.

We take theories to be sets of sentences closed under IQC-derivable consequence. Let Γ be a theory. A model is a Γ-model if it forces sentences of Γ at all nodes.

Let Γ be an intuitionistic theory.

1. We say that a formula $\varphi(\overline{x})$ of $\mathcal{L}$ is preserved under submodels of Γ-Kripke models if for any Γ-Kripke models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}$ is a submodel of $\mathfrak{B}$ and for any node α of $\boldsymbol{A}$ and for any $\overline{a}$ in A_α, if $\alpha \Vdash_{\mathfrak{B}} \varphi(\overline{a})$, then $\alpha \Vdash_{\mathfrak{A}} \varphi(\overline{a})$.
2. We say that a formula $\varphi(\overline{x})$ of $\mathcal{L}$ is preserved under extensions of Γ-Kripke models if for any Γ-Kripke models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}$ is a submodel of $\mathfrak{B}$ and for any node α of $\boldsymbol{A}$ and for any $\overline{a}$ in A_α, if $\alpha \Vdash_{\mathfrak{A}} \varphi(\overline{a})$, then $\alpha \Vdash_{\mathfrak{B}} \varphi(\overline{a})$.

We define preservation of a theory Δ under submodels (extensions) of Γ-Kripke models in a similar way.

Fact (Visser, 2001)

*Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a language $\mathcal{L}$. Then Δ is axiomatizable by **semipositive** sentences over Γ if and only if Δ is preserved under submodels of Γ-Kripke models.*

A semipositive formula is one of which each implicational subformula has an atomic antecedent.

Intuitionistic analogues of the Łos-Tarski Theorem

Let $\mathcal{E}^+$ be the set of all formulas of $\mathcal{L}$ built using only connectives $\vee$, $\wedge$ and $\exists$. We call the formulas in $\mathcal{E}^+$ **existential positive**.

Fact (Markovic, 1983)

A formula $\varphi(\overline{x})$ of $\mathcal{L}$ is intuitionistically equivalent to an existential positive formula if and only if for any Kripke model $\mathfrak{A} = ((A_\alpha)_{\alpha \in \mathbf{A}}, \leq)$, any $\alpha \in \mathbf{A}$ and any $\overline{a}$ in A_α, $\alpha \Vdash_{\mathfrak{A}} \varphi(\overline{a})$ if and only if $A_\alpha \vDash \varphi(\overline{a})$.

Theorem

A formula $\varphi(\overline{x})$ of $\mathcal{L}$ is preserved under ***extensions*** *of Kripke models if and only if it is intuitionistically equivalent to an* ***existential positive*** *formula.*

Proof Let $\varphi(\overline{x})$ be a formula of $\mathcal{L}$. If φ is intuitionistically equivalent to an existential positive formula, then it is easy to show by induction on complexity of φ that it is preserved under extensions of Kripke models. If φ is not intuitionistically equivalent to an existential positive formula, we distinguish two cases:

Case 1: There is a Kripke model $\mathfrak{A} = ((A_\alpha)_{\alpha\in\boldsymbol{A}}, \leq)$, $\alpha \in \boldsymbol{A}$ and $\overline{a}$ in A_α, such that $\alpha \Vdash_{\mathfrak{A}} \varphi(\overline{a})$ and $A_\alpha \nvDash \varphi(\overline{a})$. Let $\mathfrak{B}$ be the Kripke model obtained by putting a node β above α in $\mathfrak{A}$ and $B_\beta = A_\alpha$. We have $\mathfrak{A} \subseteq \mathfrak{B}$, $\alpha \Vdash_{\mathfrak{A}} \varphi(\overline{a})$ and $\alpha \nVdash_{\mathfrak{B}} \varphi(\overline{a})$. Thus φ is not preserved under extensions of Kripke models.

Case 2: There is a Kripke model $\mathfrak{A} = ((A_\alpha)_{\alpha\in\boldsymbol{A}}, \leq)$, $\alpha \in \boldsymbol{A}$ and $\overline{a}$ in A_α, such that $\alpha \nVdash_{\mathfrak{A}} \varphi(\overline{a})$ and $A_\alpha \vDash \varphi(\overline{a})$. Let $\mathfrak{B}$ be the Kripke model with only one node A_α. We have $\mathfrak{B} \subseteq \mathfrak{A}$, $\alpha \Vdash_{\mathfrak{B}} \varphi(\overline{a})$ and $\alpha \nVdash_{\mathfrak{A}} \varphi(\overline{a})$. So φ is not preserved under extensions of Kripke models. □

Definition

Let $\mathfrak{A} = ((A_\alpha)_{\alpha \in \boldsymbol{A}}, \leq_{\boldsymbol{A}})$ and $\mathfrak{B} = ((B_\alpha)_{\alpha \in \boldsymbol{B}}, \leq_{\boldsymbol{B}})$ be Kripke models. Then $\mathfrak{A}$ is a **submodel** of $\mathfrak{B}$, written $\mathfrak{A} \subseteq^2 \mathfrak{B}$, if and only if

1. $\boldsymbol{A} = \boldsymbol{B}$,
2. For all $\alpha \in \boldsymbol{A}$, the structure A_α is a classical submodel of B_α.

We also say that $\mathfrak{B}$ is an **extension** of $\mathfrak{A}$.

Proposition

Let $\mathfrak{A} = ((A_\alpha)_{\alpha \in \boldsymbol{A}}, \leq)$ *and* $\mathfrak{B} = ((B_\alpha)_{\alpha \in \boldsymbol{B}}, \leq)$ *be Kripke models such that* $\mathfrak{A} \subseteq^2 \mathfrak{B}$. *Let* $\varphi(\overline{x})$ *be a quantifier-free formula,* $\alpha \in \boldsymbol{A}$ *and* $\overline{a} \in A_\alpha$. *Then*

$$\alpha \Vdash_{\mathfrak{A}} \varphi(\overline{a}) \iff \alpha \Vdash_{\mathfrak{B}} \varphi(\overline{a}).$$

Proposition

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$$\alpha \Vdash_{\mathfrak{A}} \varphi(\overline{a}) \iff \alpha \Vdash_{\mathfrak{B}} \varphi(\overline{a}).$$

Proof. Induction on the complexity of φ, for all α simultaneously. □

Definition

We define two classes of formulas $\mathcal{U}$ and $\mathcal{E}$ as follows:

$$\begin{array}{ll}
\mathcal{A}\mathrm{t} \subseteq \mathcal{U}, & \mathcal{A}\mathrm{t} \subseteq \mathcal{E}, \\
\varphi, \varphi' \in \mathcal{U} \ \Rightarrow\ \varphi \vee \varphi', \varphi \wedge \varphi' \in \mathcal{U}, & \psi, \psi' \in \mathcal{E} \ \Rightarrow\ \psi \vee \psi', \psi \wedge \psi' \in \mathcal{E}, \\
\psi \in \mathcal{E}, \varphi \in \mathcal{U} \ \Rightarrow\ \psi \to \varphi \in \mathcal{U}, & \varphi \in \mathcal{U}, \psi \in \mathcal{E} \ \Rightarrow\ \varphi \to \psi \in \mathcal{E}, \\
\varphi \in \mathcal{U} \ \Rightarrow\ \forall x \varphi \in \mathcal{U}, & \psi \in \mathcal{E} \ \Rightarrow\ \exists x \psi \in \mathcal{E}.
\end{array}$$

Theorem

Let $\Gamma \subseteq \Delta$ be two intuitionistic theories. Then:

- Δ *is preserved under taking* Γ*-Kripke submodels if and only if* Δ *is axiomatizable by* $\mathcal{U}$*-sentences over* Γ.
- Δ *is preserved under taking* Γ*-Kripke extensions if and only if* Δ *is axiomatizable by* $\mathcal{E}$*-sentences over* Γ.

Definition

Let $\mathfrak{A} = ((A_\alpha)_{\alpha \in \boldsymbol{A}}, \leq_{\boldsymbol{A}})$ and $\mathfrak{B} = ((B_\alpha)_{\alpha \in \boldsymbol{B}}, \leq_{\boldsymbol{B}})$ be Kripke models. Then $\mathfrak{A}$ is a **submodel** of $\mathfrak{B}$, written $\mathfrak{A} \subseteq^3 \mathfrak{B}$, if and only if

1. $\boldsymbol{A}$ is a subset of $\boldsymbol{B}$ and $\leq_{\boldsymbol{A}} = \leq_{\boldsymbol{B}} \restriction \boldsymbol{A}$,
2. For all $\alpha \in \boldsymbol{A}$, the structure A_α is a classical submodel of B_α.

We also say that $\mathfrak{B}$ is an **extension** of $\mathfrak{A}$.

Fact (Ellison et al., 2007)

Let $\Gamma \subseteq \Delta$ be intuitionistic theories over a language $\mathcal{L}$. Then Δ is axiomatizable by universal semipositive sentences over Γ if and only if Δ is preserved under submodels of Kripke Γ-models.

The class of **universal semipositive** formulas $\mathcal{U} \subseteq \mathcal{L}$ is defined inductively as follows:

$$\varphi \in \mathcal{A}\mathrm{t} \Rightarrow \varphi \in \mathcal{U}, \qquad \varphi, \psi \in \mathcal{U} \Rightarrow \varphi \vee \psi, \varphi \wedge \psi \in \mathcal{U},$$

$$\varphi \in \mathcal{A}\mathrm{t}, \psi \in \mathcal{U} \Rightarrow \varphi \to \psi \in \mathcal{U}, \qquad \varphi \in \mathcal{U} \Rightarrow \forall x \varphi \in \mathcal{U}.$$

Intuitionistic Universal and Existential Closures

Let A and B be classical $\mathcal{L}$-structures. By $A \Lleftarrow_{\exists_1} B$ we mean that $A \subseteq B$ and for every $\exists_1$ formula $\varphi(\overline{x})$ of $\mathcal{L}$ and every tuple $\overline{a}$ of elements of A, if $B \vDash \varphi(\overline{a})$ then $A \vDash \varphi(\overline{a})$. The notion $A \Rrightarrow_{\forall_1} B$ is defined in the same way. Obviously, $A \Lleftarrow_{\exists_1} B$ and $A \Rrightarrow_{\forall_1} B$ are equivalent.

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Fact

Let T be a classical theory in $\mathcal{L}$. The following are equivalent:

- *T is axiomatizable by $\forall_2$-sentences.*
- *If $A \Leftarrow_{\exists_1} B$ and $B \vDash T$, then $A \vDash T$.*
- *If $A^1 \subseteq A^2 \subseteq A^3 \subseteq \cdots$ is a chain of models of T, then $\bigcup_i A^i$ is also a model of T, i.e. T is preserved in unions of chains of models of T.*

Definition

Let $\Phi, \Psi \subseteq \mathcal{L}$ be sets of formulas. Let $\mathfrak{A} = ((A_\alpha)_{\alpha \in F}, \leq)$ and $\mathfrak{B} = ((B_\alpha)_{\alpha \in F}, \leq)$ be Kripke models over $\mathcal{L}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$.

- We write $\mathfrak{A} \Lleftarrow_\Phi \mathfrak{B}$, if for all $\alpha \in F$ and $\varphi \in \Phi(A_\alpha)$,
$$\alpha \Vdash_{\mathfrak{B}} \varphi \quad \Rightarrow \quad \alpha \Vdash_{\mathfrak{A}} \varphi.$$
- We write $\mathfrak{A} \Rrightarrow_\Psi \mathfrak{B}$, if for all $\alpha \in F$ and $\psi \in \Psi(A_\alpha)$,
$$\alpha \Vdash_{\mathfrak{A}} \psi \quad \Rightarrow \quad \alpha \Vdash_{\mathfrak{B}} \psi.$$
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Let $\Phi, \Psi \subseteq \mathcal{L}$ be sets of formulas. Let $\mathfrak{A} = ((A_\alpha)_{\alpha\in F}, \leq)$ and $\mathfrak{B} = ((B_\alpha)_{\alpha\in F}, \leq)$ be Kripke models over $\mathcal{L}$ such that $\mathfrak{A} \subseteq \mathfrak{B}$.

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- We write $\mathfrak{A} \,{}_\Phi\!\Longleftrightarrow_\Psi \mathfrak{B}$, if $\mathfrak{A} \Lleftarrow_\Phi \mathfrak{B}$ and $\mathfrak{A} \Rrightarrow_\Psi \mathfrak{B}$.

Note that $\mathfrak{A} \Lleftarrow_{\mathcal{E}} \mathfrak{B}$ and $\mathfrak{A} \Rrightarrow_{\mathcal{U}} \mathfrak{B}$ are not generally equivalent for Kripke models.

Definition

Let $\Phi, \Psi \subseteq \mathcal{L}$ be sets of formulas. Let the sets $\mathcal{U}(\Phi, \Psi)$ of universal-(Φ, Ψ) formulas and $\mathcal{E}(\Phi, \Psi)$ of existential-(Φ, Ψ) formulas be the smallest sets such that:

$$\Phi \subseteq \mathcal{U}(\Phi, \Psi),$$
$$\varphi, \varphi' \in \mathcal{U}(\Phi, \Psi) \Rightarrow \varphi \vee \varphi', \varphi \wedge \varphi' \in \mathcal{U}(\Phi, \Psi),$$
$$\psi \in \mathcal{E}(\Phi, \Psi), \varphi \in \mathcal{U}(\Phi, \Psi) \Rightarrow \psi \rightarrow \varphi \in \mathcal{U}(\Phi, \Psi),$$
$$\varphi \in \mathcal{U}(\Phi, \Psi) \Rightarrow \forall x \varphi \in \mathcal{U}(\Phi, \Psi),$$

$$\Psi \subseteq \mathcal{E}(\Phi, \Psi),$$
$$\psi, \psi' \in \mathcal{E}(\Phi, \Psi) \Rightarrow \psi \vee \psi', \psi \wedge \psi' \in \mathcal{E}(\Phi, \Psi),$$
$$\varphi \in \mathcal{U}(\Phi, \Psi), \psi \in \mathcal{E}(\Phi, \Psi) \Rightarrow \varphi \rightarrow \psi \in \mathcal{E}(\Phi, \Psi),$$
$$\psi \in \mathcal{E}(\Phi, \Psi) \Rightarrow \exists x \psi \in \mathcal{E}(\Phi, \Psi).$$

Generalized Preservation Theorems

Theorem

Let $\Gamma \subseteq \Delta$ *be intuitionistic theories over* $\mathcal{L}$*, and let* $\Phi, \Psi \subseteq \mathcal{L}$ *such that* $\mathcal{A}\mathrm{t} \subseteq \Phi$ *and* $\mathcal{A}\mathrm{t} \subseteq \Psi$*. The following are equivalent:*

- Δ *is axiomatizable by* $\mathcal{U}(\Phi, \Psi)$*-sentences over* Γ*.*
- *For all Kripke models* $\mathfrak{A} \Vdash \Gamma$ *and* $\mathfrak{B} \Vdash \Delta$*, if* $\mathfrak{A} \;{}_{\Phi}{\Longleftrightarrow}_{\Psi}\; \mathfrak{B}$*, then* $\mathfrak{A} \Vdash \Delta$*.*

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- *Δ is axiomatizable by $\mathcal{E}(\Phi, \Psi)$-sentences over Γ.*
- *For all Kripke models $\mathfrak{A} \Vdash \Delta$ and $\mathfrak{B} \Vdash \Gamma$, if $\mathfrak{A} \;{}_{\Phi}\!\!\Longleftrightarrow_{\Psi} \mathfrak{B}$, then $\mathfrak{B} \Vdash \Delta$.*

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Definition

We define a class $\mathcal{U}_2^4 \subseteq \mathcal{L}$ of formulas inductively as follows:

$\varphi \in \mathcal{E} \Rightarrow \varphi \in \mathcal{U}_2^4$, $\qquad \varphi, \psi \in \mathcal{U}_2^4 \Rightarrow \varphi \vee \psi, \varphi \wedge \psi \in \mathcal{U}_2^4$,

$\varphi \in \mathcal{U}, \psi \in \mathcal{U}_2^4 \Rightarrow \varphi \to \psi \in \mathcal{U}_2^4$, $\qquad \varphi \in \mathcal{U}_2^4 \Rightarrow \forall x\varphi \in \mathcal{U}_2^4$.

Generalized Preservation Theorems

Corollary

Let $\Gamma \subseteq \Delta$ *be intuitionistic theories. The following are equivalent:*

- Δ *is axiomatizable by* $\mathcal{U}_2^1$*-sentences over* Γ.
- *For all Kripke models* $\mathfrak{A} \Vdash \Gamma$ *and* $\mathfrak{B} \Vdash \Delta$, *if* $\mathfrak{A} \Lleftarrow_{\mathcal{E}} \mathfrak{B}$, *then* $\mathfrak{A} \Vdash \Delta$.

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Corollary

Let $\Gamma \subseteq \Delta$ *be intuitionistic theories. The following are equivalent:*

- Δ *is axiomatizable by* $\mathcal{U}_2^2$*-sentences over* Γ.
- *For all Kripke models* $\mathfrak{A} \Vdash \Gamma$ *and* $\mathfrak{B} \Vdash \Delta$, *if* $\mathfrak{A} \Rrightarrow_{\mathcal{U}} \mathfrak{B}$, *then* $\mathfrak{A} \Vdash \Delta$.

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Corollary

Let $\Gamma \subseteq \Delta$ *be intuitionistic theories. The following are equivalent:*

- Δ *is axiomatizable by* $\mathcal{U}_2^3$*-sentences over* Γ.
- *For all Kripke models* $\mathfrak{A} \Vdash \Gamma$ *and* $\mathfrak{B} \Vdash \Delta$*, if* $\mathfrak{A} \, {}_{\mathcal{E}}\!\Longleftrightarrow_{\mathcal{U}} \mathfrak{B}$*, then* $\mathfrak{A} \Vdash \Delta$.

Chains of Kripke Models

Let $\mathfrak{A}^1 \subseteq \mathfrak{A}^2 \subseteq \mathfrak{A}^3 \subseteq \cdots$ be a chain of Kripke models with the same frame F. For every $\alpha \in F$, we have $A^1_\alpha \subseteq A^2_\alpha \subseteq A^3_\alpha \subseteq \cdots$ (submodel in the classical sense). So, using the classical construction, we can define $M_\alpha = \bigcup_i A^i_\alpha$ as usual. Clearly, M_α is a weak substructure of M_β whenever $\alpha \leq \beta$.

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Let $\mathfrak{A}^1 \subseteq \mathfrak{A}^2 \subseteq \mathfrak{A}^3 \subseteq \cdots$ be a chain of Kripke models with the same frame F. We define $\bigcup_i \mathfrak{A}^i$ to be Kripke model $((M_\alpha)_{\alpha \in F}, \leq)$, where $M_\alpha = \bigcup_i A^i_\alpha$.

Let $\mathfrak{A}^1 \subseteq \mathfrak{A}^2 \subseteq \mathfrak{A}^3 \subseteq \cdots$ be a chain of Kripke models with the same frame F. For every $\alpha \in F$, we have $A^1_\alpha \subseteq A^2_\alpha \subseteq A^3_\alpha \subseteq \cdots$ (submodel in the classical sense). So, using the classical construction, we can define $M_\alpha = \bigcup_i A^i_\alpha$ as usual. Clearly, M_α is a weak substructure of M_β whenever $\alpha \leq \beta$.

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Let $\mathfrak{A}^1 \subseteq \mathfrak{A}^2 \subseteq \mathfrak{A}^3 \subseteq \cdots$ be a chain of Kripke models with the same frame F. We define $\bigcup_i \mathfrak{A}^i$ to be Kripke model $((M_\alpha)_{\alpha \in F}, \leq)$, where $M_\alpha = \bigcup_i A^i_\alpha$.

It is easy to see that, for every $n \in \mathbb{N}$, we have $\mathfrak{A}^n \subseteq \bigcup_i \mathfrak{A}^i$.

The following example shows that none of the classes $\mathcal{U}_2^1$, $\mathcal{U}_2^2$ and $\mathcal{U}_2^3$ is generally preserved in unions of chains.

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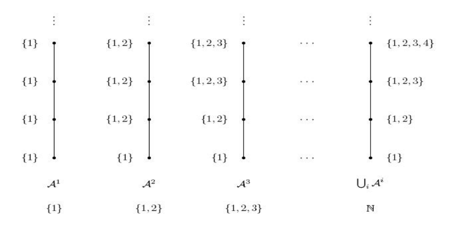
Example

Let $\mathcal{L}_1 = \{R\}$ be a first order language containing only one unary predicate symbol R. We define a chain $\mathfrak{A}^1 \subseteq \mathfrak{A}^2 \subseteq \mathfrak{A}^3 \subseteq \cdots$ of Kripke models of $\mathcal{L}_1$ with the same frame ω as follows (the domains are fixed and written under each Kripke model and the interpretation of R in each node is shown beside it).



Chains of Kripke Models

The sentence $\varphi := \neg\neg\forall x R(x)$ is an $\mathcal{U}$-sentence. The formula φ is forced in all elements of the chain but is not forced in the union of the chain. Since each of the classes $\mathcal{U}_2^1$, $\mathcal{U}_2^2$ and $\mathcal{U}_2^3$ contain $\mathcal{U}$, they are not preserved in unions of chains.

Chains of Kripke Models

Theorem

Let $\Gamma \subseteq \Delta$ be two intiutionistic theories over $\mathcal{L}$. Suppose that Δ is axiomatizable by $\mathcal{U}_2^4$-sentences over Γ. Then for each chain $\mathfrak{A}^1 \subseteq \mathfrak{A}^2 \subseteq \mathfrak{A}^3 \subseteq \cdots$ of Kripke models of Δ, if $\bigcup_i \mathfrak{A}^i \Vdash \Gamma$, then $\bigcup_i \mathfrak{A}^i \Vdash \Delta$, i.e. Δ is preserved in unions of chains of Kripke models of Δ.

Below we bring a natural definition of the notion *elementary submodel*.

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Definition

Let $\mathfrak{A} = ((A_\alpha)_{\alpha \in F}, \leq)$ and $\mathfrak{B} = ((B_\alpha)_{\alpha \in F}, \leq)$ be two Kripke models. We say that $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$, denoted $\mathfrak{A} \preccurlyeq \mathfrak{B}$, if:

- $\mathfrak{A} \subseteq \mathfrak{B}$,
- For any formula $\varphi(\overline{x})$, $\alpha \in F$ and $\overline{a} \in A_\alpha$, $\alpha \Vdash_{\mathfrak{A}} \varphi(\overline{a})$ if and only if $\alpha \Vdash_{\mathfrak{B}} \varphi(\overline{a})$.

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In this case we also say that $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$.

Theorem

Let $\mathfrak{A}^1 \preccurlyeq \mathcal{A}^2 \preccurlyeq \mathfrak{A}^3 \preccurlyeq \cdots$ *be an elementary chain of Kripke models with the same frame* F*. Then, for every* $n \in \mathbb{N}$*,* $\mathfrak{A}^n \preccurlyeq \bigcup_i \mathfrak{A}^i$*.*

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Thank You