Probability (Modal) Logic

Reihane Zoghifard

Institute for research in fundamental sciences (IPM)

IAL

August 19, 2019

Contents

Contents

There are two approaches to belief types in game theory:

1. Explicit description consist of a hierarchy of beliefs which satisfy the coherence requirement that different levels of beliefs of every agent do not contradict one another.
2. Implicit description introduced by Harsanyi in 1960's for games with incomplete information played by Bayesian players. Belief types proposed by a probability distribution over the set of events.

Type Space

A type space is a triple $\mathcal{S} = (\Omega, \mathcal{A}, (T_i)_{i \in I})$ where:

- Ω is a set of states.
- $\mathcal{A}$ is a σ-algebra over Ω.
- For each agent $i \in I$, $T_i : \Omega \times \mathcal{A} \to [0, 1]$ is a function such that
  - $T_i(w)(.)$ is a probability measure, for each $w \in \Omega$,
  - $T_i(.)(E)$ is measurable i.e. $\{w \in \Omega \mid T_i(w)(E) \geq r\} \in \mathcal{A}$ for each $E \in \mathcal{A}$ and $r \in \mathbb{Q} \cap [0, 1]$.

In 1990, Fagin and et al. introduced a syntax containing the linear combination of probabilities, such as:

$$a_1 w_i(\phi_1) + \cdots + a_k w_i(\phi_k) \geq c$$

- FAGIN, R., HALPERN, J. Y., AND MEGIDDO, N. A logic for reasoning about probabilities. *Information and computation 87*, 1-2 (1990), 78–128.

Syntax

In 1999, Aumann introduced a simpler syntax:

Let I be a set of agents.

$$\phi := p \mid \neg\phi \mid \phi \wedge \phi \mid L^i_r\phi \ \ (r \in \mathbb{Q} \cap [0,1])$$

$L^i_r\phi$: "agent i assigns probability at least r to ϕ".

$M^i_r\phi = L^i_{1-r}\neg\phi$: "agent i assigns probability at most r to ϕ".

A model is a tuple $\mathfrak{M} = (\Omega, \mathcal{A}, (T_i)_{i \in I}, v)$ where:

- $(\Omega, \mathcal{A}, (T_i)_{i \in I})$ is a type space.
- v is a valuation function such that $v(p) \in \mathcal{A}$ for each $p \in \mathcal{P}$.

The satisfaction is defined as follows:

$$\mathfrak{M}, w \models L_r^i \phi \quad \text{iff} \quad T_i(w)\{w' \in W \mid \mathfrak{M}, w' \models \phi\} \geq r.$$

Probability logic is not compact. For example:

$$T = \{L_{\frac{1}{2}-\frac{1}{4^{n+1}}}\, p \mid n \in \mathbb{N}\} \cup \{\neg L_{\frac{1}{2}}\, p\}.$$

Contents

Aumann's System Σ

- A0 Any axiomatization of the propositional calculus
- A1 $L_0\phi$
- A2 $L_r\top$
- A3 $L_r(\phi \wedge \psi) \wedge L_s(\phi \wedge \neg\psi) \rightarrow L_{r+s}\phi, \quad r+s \leq 1$
- A4 $\neg L_r(\phi \wedge \psi) \wedge \neg L_s(\phi \wedge \neg\psi) \rightarrow \neg L_{r+s}\phi, \quad r+s \leq 1$
- A5 $L_r\phi \rightarrow \neg L_s\neg\phi \quad r+s > 1$
- A6 If $\vdash \phi \leftrightarrow \psi$ then $\vdash L_r\phi \leftrightarrow L_r\psi$.

Aumann's system is sound but incomplete.

The following schemata are not entailed by Σ:

A13 $M_r(\phi \wedge \psi) \wedge M_s(\phi \wedge \neg\psi) \rightarrow M_{r+s}\phi, \quad r + s \leq 1.$

A14 $M_r(\phi \wedge \psi) \wedge \neg L_s(\phi \wedge \neg\psi) \rightarrow \neg L_{r+s}\phi, \quad r + s \leq 1.$

$\Sigma^+ = \Sigma + (B)$

(B) If $((\phi_1, \dots, \phi_m) \leftrightarrow (\psi_1, \dots, \psi_n))$, then

$$((\bigwedge_{i=1}^{m} L_{r_i}\phi_i) \wedge (\bigwedge_{j=2}^{n} M_{s_j}\psi_j) \rightarrow L_{(r_1+\cdots+r_m)-(s_2+\cdots+s_m)}\psi_1)$$

for $m, n \geq 1$ and $(r_1 + \cdots + r_m) - (s_2 + \cdots + s_m) \in [0, 1]$.

$\phi^{(k)}$ refer to either $\bigvee_{1\leq L-1<\cdots<l_k\leq m}(\phi_{l_1} \wedge \cdots \wedge \phi_{l_k})$ or to $\perp$ whenever $k > m$.
$(\phi_1, \dots, \phi_m) \leftrightarrow (\psi_1, \dots, \psi_n)$ refer to $\bigwedge_{k=1}^{\max(m,n)} \phi^{(k)} \leftrightarrow \psi^{(k)}$.

$\Sigma^+ = \Sigma + (B)$

(B) If $((\phi_1, \ldots, \phi_m) \leftrightarrow (\psi_1, \ldots, \psi_n))$, then

$$((\bigwedge_{i=1}^{m} L_{r_i}\phi_i) \wedge (\bigwedge_{j=2}^{n} M_{s_j}\psi_j) \rightarrow L_{(r_1+\cdots+r_m)-(s_2+\cdots+s_m)}\psi_1)$$

for $m, n \geq 1$ and $(r_1 + \cdots + r_m) - (s_2 + \cdots + s_m) \in [0, 1]$.

Theorem

Σ^+ is sound and complete.

- HEIFETZ, A. and P. MONGIN, 'Probability logic for type spaces', Games and economic behavior, 2001.

$\Sigma_+ = \Sigma + (ARCH)$

(ARCH) If $\vdash \phi \to L_s\psi$ for all $s < r$, then $\vdash \phi \to L_r\psi$.

Theorem

Σ_+ is sound and complete.

- ZHOU, C, *Complete Deductive Systems for Probability Logic with Application to Harsanyi Type Spaces*, PhD thesis, 2007.

Moss's conjecture:

Theorem (Zhou)

For formulas ϕ and ψ, we can constructively find a sufficiently small rational number ϵ, which depends only on the depth, the accuracy and the number of propositional letters of ϕ and ψ such that

$$(ARCH^f): \vdash_{\Sigma_+} \phi \to L_{r-\epsilon}\psi \ \Rightarrow \vdash_{\Sigma_+} \phi \to L_r\psi.$$

Theorem

For any formula ϕ, we have $\vdash_{\Sigma_+} \phi$ iff $\vdash_{\Sigma_+^f} \phi$.

Harsanyi type spaces

A type space $(\Omega, \mathcal{A}, (T_i)_{i \in I})$ is a Harsanyi type when $T_i(w)(E) = 1$ for all $w \in \Omega$ and $E \in \mathcal{A}$ such that $\{w' \mid T_i(w) = T_i(w')\} \subseteq E$.

$\Sigma_H = \Sigma + 4 + 5$

4 $L_r^i \phi \rightarrow L_1^i L_r^i \phi$

5 $\neg L_r^i \phi \rightarrow L_1^i \neg L_r^i \phi$

Knowledge and belief

$$\Sigma_{HK} = \Sigma_H + (S_5)_K + \{H_1, H_2, H_3\}$$

H_1 $L_r\phi \rightarrow KL_r\phi$

H_2 $\neg L_r\phi \rightarrow K\neg L_r\phi$

H_3 $K\phi \rightarrow L_1\phi$

Strong completeness:

To prove the strong completeness we have to add the following rules to Σ_+:

- (CAR) $\Gamma \vdash \varphi$ implies $\{L_r\psi \mid \psi \in \bigwedge_\omega \Gamma\} \vdash L_r\varphi$,

  where $\bigwedge_\omega \Gamma$ is the set of conjunctions of finite subsets of Γ.
- Lindenbaum property

- GOLDBLATT, R. Deduction systems for coalgebras over measurable spaces. *Journal of Logic and Computation 20*, 5 (2010), 1069–1100.

Strong completeness with respect to Markov processes:

$\Sigma + R_2$

R_2 $\{L_{r_1 \ldots r_k s}\phi \mid s < r\} \vdash L_{r_1 \ldots r_k r}\phi$

- GOLDBLATT, R. KOZEN, D., MARDARE, R., AND PANANGADEN, P. Strong completeness for markovian logics. In *International Symposium on Mathematical Foundations of Computer Science* (2013), Springer, pp. 655–666.

A model $\mathfrak{M} = (\Omega, \mathcal{A}, T, v)$ is finitely additive if $T(w)(.)$ is a finitely additive probability measure for some $w \in \Omega$.

Theorem

Σ_+ *is strongly complete with respect to the class of finitely additive models.*

- ZHOU, C. Probability logic of finitely additive beliefs. *Journal of Logic, Language and Information 19*, 3 (2010), 247–282.

Contents

Probability logic is not compact. For example:

$$T = \{L_{\frac{1}{2}-\frac{1}{4n+1}}\, p \mid n \in \mathbb{N}\} \cup \{\neg L_{\frac{1}{2}}\, p\}.$$

There are a number of sources of non-compactness:

- σ-additivity,
- Archimedean property of rational numbers,
- explicit use of negation.

A formula ϕ is a *basic* formula if it has one of the following syntax forms:

$$\phi := p \mid \neg p \mid \phi \wedge \phi \mid \phi \vee \phi \mid L_r\phi \quad (r \in \mathbb{Q} \cap [0, 1]).$$

A formula ϕ is a *positive* formula if it has one of the following syntax forms:

$$\phi := p \mid \neg p \mid \phi \wedge \phi \mid \phi \vee \phi \mid L_r\phi \mid M_r\phi \; (r \in \mathbb{Q} \cap [0, 1]).$$

Ultraproduct

Let $\langle \mathfrak{M}_i = (\Omega_i, \mathcal{A}_i, T_i, v_i) : i \in I\rangle$ be a family of probability models and U be an ultrafilter over I. The ultraproduct of $\mathfrak{M}_i$s over U is a model $\mathfrak{M} = \prod_U \mathfrak{M}_i = (\Omega_U, \mathcal{A}_U, T_U, v_U)$ where

- $\Omega_U = \prod_U \Omega_i$,
- $\mathcal{A}_U$ is a σ-algebra generated by the set of all $[(A_i)]$, where $A_i \in \mathcal{A}_i$ and $[(A_i)] = \{[(a_i)] \in W \mid \{i \in \mathbb{N} \mid a_i \in A_i\} \in U\}$,
- T_U is a measurable function induced by the premeasure $T' : \prod_U \Omega_i \times \mathcal{A} \to [0, 1]$

$$T'([(w_i)])([(A_i)]) = \lim_U T_i(w_i)(A_i).$$

- $[(w_i)] \in v_U(p)$ if and only if $\{i \in I \mid w_i \in v_i(p)\} \in U$.

Łoś's Theorem for Probability Logic

Let $\langle \mathfrak{M}_i : i \in I \rangle$ be a family of probability models and U be a non-principal ultrafilter over I. Then $\{i \in I \mid \mathfrak{M}_i, w_i \vDash \phi\} \in U$ implies $\prod_U \mathfrak{M}_i, [(w_i)] \vDash \phi$, for every basic formula ϕ.

M. Pourmahdian and R. Zoghifard

Suppose that Γ is a set of basic formulas. Γ is satisfiable if and only if it is finitely satisfiable.

Example:

Let

$$\Sigma = \{\ M_0(M_0 p \vee L_1 p)\ \} \cup \{\ M_{\frac{1}{2}}(L_{\frac{1}{2^i}} p \wedge M_{1-\frac{1}{2^i}} p) \mid i \in \mathbb{N}\}.$$

Σ is finitely satisfiable but it is not satisfiable in any probability model.

Σ has a finitely additive model.

Let $\mathfrak{M} = (\mathbb{N}, \mathcal{P}(\mathbb{N}), T, v)$ be a probability model where $v(p) = \{0\}$. For each $n \neq 0$,

$$T(n)(\{x\}) = \begin{cases} \frac{1}{2^n} & \text{if } 0 \leq x \leq 2^n - 1 \\ 0 & \text{if } x > 2^n - 1 \end{cases}$$

Let U be a non-principal ultrafilter over ω and define $T(0)$ as follows:

$$T(0)(X) = \begin{cases} 1 & \text{if } X \in U \\ 0 & \text{if } X \notin U \end{cases}$$

$T(0)$ is a finitely additive and is not a σ-additive measure.

$\mathfrak{M}, 0 \models \Sigma$.

Finitely Additive Models

M. Pourmahdian and R. Zoghifard

Let Γ be a set of positive formulas which is finitely satisfiable. Then Γ has a finitely additive model.

Contents

Lindström Theorem for First-Order Logic

Lindström, 1969

Any abstract logic extending first-order logic with compactness and the Löwenheim-Skolem property has the same expressive power as first-order logic.

Lindström Theorem for Modal Logic

De Rijke, 1995

Any abstract logic extending modal logic with the finite depth and bisimulation invariance properties does not have more expressive power than modal logic.

Van Benthem, 2006

An abstract logic extending modal logic is equivalent to it if and only if it satisfies compactness and the relativization property and is invariant under bisimulations.

- OTTO, M. and R. PIRO, ‘A Lindström characterisation of the guarded fragment and of modal logic with a global modality’, *Advances in Modal Logic*, 2008.
- VAN BENTHEM, J., B. TEN CATE, and J.A. VÄÄNÄNEN, ‘Lindström theorems for fragments of first-order logic’, *Logical Methods in Computer Science* 5, 2009.
- KURZ, A. and Y. VENEMA, ‘Coalgebraic Lindström Theorem’, Advances in Modal Logic, Moscow 2010.
- ENQVIST, S., ‘A General Lindström Theorem for Some Normal Modal Logics’, *Logica Universalis* 7: 233, 2013.
- ENQVIST, S., ‘A new coalgebraic Lindström theorem’, *Journal of Logic and Computation* 26(5):1541–1566, 2016.

Filtration

- Let $\mathcal{L}(P, q, d)$ be the set of formulas which their propositions are among the set P, their indexes are multiples of $\frac{1}{q}$ and the depth of them is at most d.
- For any probability model $\mathfrak{M}$ and a triple (P, q, d) a filtered probability model $\mathfrak{M}(P, q, d)$ is any model satisfying the following properties:
    - $\Omega_{(P,q,d)} = \{[w]_{(P,q,d)} \mid w \in \Omega\}$,
    - $\mathcal{A}_{(P,q,d)} = \mathcal{P}(\Omega_{(P,q,d)})$.
    - For each $[w]_{(P,q,d)} \in \Omega_{(P,q,d)}$, we have $T_{(P,q,d)}([w]_{(P,q,d)}) = T(w'')$ for some $w'' \in [w]_{(P,q,d)}$.
    - $v_{(P,q,d)}(p) = \{[w] \mid w \in v(p)\}$.

Proposition

Let $\mathfrak{M}(P, q, d)$ be a filtered model of $\mathfrak{M}$ through $\mathcal{L}(P, q, d)$. Then for any $w \in \Omega$ and any $\varphi \in \mathcal{L}(P, q, d)$,

$$\mathfrak{M}, w \vDash \varphi \text{ if and only if } \mathfrak{M}(P, q, d), [w]_{(P,q,d)} \vDash \varphi.$$

Suppose that L is an abstract logic.

- L has the finite depth property if for each formula $\varphi \in L$ there is an $n \in \mathbb{N}$ such that for any probability model $\mathfrak{M}$ we have $\mathfrak{M}, w \vDash \varphi$ if and only if $\mathfrak{M}_n, [w]_n \vDash \varphi$. Where $\mathfrak{M}_n$ is the filtration of $\mathfrak{M}$ to the language $\mathcal{L}(n, n, n)$.
- L is invariant under disjoint unions if for any formula $\varphi \in L$ and probability models $\mathfrak{M}$ and $\mathfrak{N}$, we have $\mathfrak{M}, w \vDash \varphi$ if and only if $\mathfrak{M} \uplus \mathfrak{N}, w \vDash \varphi$.

M. Pourmahdian and R. Zoghifard

Let L be an abstract logic containing probability logic. If L has the finite depth property and is invariant under disjoint unions, then L is equivalent by PL.

Thank you for your attention.