*Iranian Association for Logic*

# First-Order Interpolation Derived from Propositional Interpolation

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**Abstract:**

Ever since Craig's seminal paper on interpolation [3], interpolation properties have been recognized as important properties of logical systems. Recall that a logic $L$ has *interpolation* if whenever $A \to B$ holds in $L$ there exists a formula $I$ in the common language of $A$ and $B$ such that $A \to I$ and $I \to B$ hold in $L$.

Propositional interpolation properties can be determined and classified with relative ease using the ground-breaking results of Maksimova cf. [7, 6, 5]. This approach is based on an algebraic analysis of the logic in question. In contrast first-order interpolation properties are notoriously hard to determine, even for logics where propositional interpolation is more or less obvious. For example it is unknown whether $\mathrm{G}^{\mathrm{QF}}_{[0,1]}$ (first-order infinitely-valued Gödel logic) interpolates (cf. [1]) and even for $\mathrm{MC}^{\mathrm{QF}}$, the logic of constant domain Kripke frames of three worlds with two top worlds (an extension of MC), interpolation proofs are very hard cf. Ono [8]. This situation is due to the lack of an adequate algebraization of non-classical first-order logics. In this paper we present a proof theoretic methodology to reduce first-order interpolation to propositional interpolation:

$$\left.\begin{array}{r}\text{existence of suitable skolemizations } + \\ \text{existence of Herbrand expansions } + \\ \text{propositional interpolation}\end{array}\right\} \Rightarrow \begin{array}{c}\text{first-order} \\ \text{interpolation.}\end{array}$$

The construction of the first-order interpolant from the propositional interpolant follows this procedure:

1. Develop a validity equivalent skolemization replacing all strong quantifiers[3] in the valid formula $A \to B$ to obtain the valid formula $A_1 \to B_1$.
2. Construct a valid Herbrand expansion $A_2 \to B_2$ for $A_1 \to B_1$. Occurrences of $\exists x B(x)$ and $\forall x A(x)$ are replaced by suitable finite disjunctions $\bigvee B(t_i)$ and conjunctions $\bigwedge B(t_i)$, respectively.
3. Interpolate the propositionally valid formula $A_2 \to B_2$ with the propositional interpolant $I^*$: $A_2 \to I^*$ and $I^* \to B_2$ are propositionally valid.
4. Reintroduce weak quantifiers to obtain valid formulas $A_1 \to I^*$ and $I^* \to B_1$.
5. Eliminate all function symbols and constants not in the common language of $A_1$ and $B_1$ by introducing suitable quantifiers in $I^*$ (note that no Skolem functions are in the common language, therefore they are eliminated). Let $I$ be the result.
6. $I$ is an interpolant for $A_1 \to B_1$. $A_1 \to I$ and $I \to B_1$ are skolemizations of $A \to I$ and $I \to B$. Therefore $I$ is an interpolant of $A \to B$.

It is decidable if propositional lattice based finitely-values logics admit the interpolation property [2]. Consequently, it is decidable if finitely-valued first-order logics admit the interpolation property. In this paper we extend the methodology to prenex fragments where Skolemization is admissible due to the second epsilon theorem [4].

## References

1. Aguilera, J.P., Baaz, M.: Ten problems in Gödel logic. Soft Computing **21**(1), 149–152 (2017)
2. Baaz, M., Lolic, A.: First-order interpolation derived from propositional interpolation. Theor. Comput. Sci. **837**, 209–222 (2020)
3. Craig, W.: Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. The Journal of Symbolic Logic **22**(03), 269–285 (1957)
4. Hilbert, D., Bernays, P.: Grundlagen der Mathematik. (1968)
5. Maksimova, L.: Intuitionistic logic and implicit definability. Annals of Pure and Applied Logic **105**(1-3), 83–102 (2000)
6. Maksimova, L.L.: Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras. Algebra and Logic **16**(6), 427–455 (1977)
7. Maksimova, L.L.: Interpolation properties of superintuitionistic logics. Studia Logica **38**(4), 419–428 (1979)
8. Ono, H.: Model extension theorem and Craig's interpolation theorem for intermediate predicate logics. Reports on Mathematical Logic **15**, 41–58 (1983)

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**17:00 - 19:00 (THR) - 13:30 - 15:30 (GMT)**

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